NASA-CR-192750

CIRSSE-R-75

NAGW-1333

TECHNICAL REPORTS

(NASA-CR-192750) NONLINEAR DYNAMICAL MODEL AND CONTROL FOR A FLEXIBLE BEAM (Rensselaer Polytechnic Inst.) 17 p

N93-71629

Unclas

Z9/63 0153772

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NONLINEAR DYNAMICAL MODEL AND CONTROL FOR A FLEXIBLE BEAM

by

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November, 1990

CIRSSE REPORT #75

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CII 8123

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Abstract

This paper presents the derivation of the dynamical motion equations for a one-link flexible manipulator and some preliminary stability results using these equations. The most general set of equations is derived under only the assumption of Hooke's Law and negligible longitudal deformation. The equations simplify first by further assuming small bending deformations, and then small bending and rotational velocities. The natural modes of the linearized partial differential equation are solved exactly, and they are used to discretize the nonlinear small–bending equation. The resulting equation has the same structure as the dynamical equation for rigid robots. An energy Lyapunov function method is proposed for the stability analysis. Excellent agreement between the analytic prediction and experimental results for the modal frequencies is also reported.

Keywords

Flexible manipulator; flexible beam; nonlinear dynamical model; energy Lyapunov analysis; Hamilton's principle.

1 Introduction

Research effort on the modeling and control of flexible manipulators has increased dramatically in the recent years. This is motivated in part by the hope for higher speed, less weight, and better energy consumption offered by such mechanisms. These issues are most relevant in the context of space structure construction and operation.

Many dynamical models have been proposed for one–link flexible manipulators in the past; ranging from the distributed parameter linearized model in [1, 2, 3, 4, 5, 6] to some recent nonlinear models such as the distributed model in [7] and the finite, discretized models in [8, 9]. All of the above are based on the small deformation assumption, and there were some confusion about the boundary condition (such as in [1, 4, 5]). This paper derives the dynamical motion equation under the mild assumption of Hooke's Law (between stress and strain) and negligible elongation. The small deformation assumption is then applied to simplify the model to the form as that in [7]. Finally, the hub and bending velocities are further assumed small, and we obtain a set of linearized equations. These equations are the same as that in [3, 6], but the integral term in the equation is removed, resulting in a much simplified set of equations.

The natural modes of the linearized equation can be derived exactly. As pointed out in [6] (and also alluded to in [10]), the characteristic equation for the modal frequency is a linear combination of the characteristic equations of the clamped–free case and the pinned–free case; whether it is closer to one case versus the other depends on the hub inertia. The set of natural modes is then used to discretize the nonlinear equation of motion under the small bending assumption. The resulting equation is of the same form as the robotic equation of motion except only one actuator is present for the infinite degrees of freedom. Through a passivity analysis based on an energy–motivated Lyapunov function, we show that the proportional plus derivative feedback of the rotational angle is globally asymptotically stabilizing if a mild detectability and stabilizability condition is satisfied.

Some experimental results are also included, showing the excellent agreement between the analytic prediction and experimental data for the first nine bending modes.

This paper is organized as follows: The large deformation model is derived by using the Hamilton's principle in section 2. Small–bending model and then the linearized model are obtained under the corresponding assumptions. The natural modes for the linearized model are obtained in section 4 and they are used to discretize the small–bending model. The energy Lyapunov function analysis is used to show the global stability of PD joint angle feedback.

2 Nonlinear Dynamical Motion Equations

The flexible manipulator to be considered is a beam of length L fixed on a hub with rotational inertia I_H in the horizontal plane as shown in Fig.1. Let (x^0, y^0) be the inertial coordinate system and (x, y) be the coordinate system that rotates with the hub.

The motion of the manipulator has been described by the angular rotation ϕ due to the hub rotation, and the horizontal displacement u and the vertical displacement w of the beam with respect to (x, y) coordinate. Clearly, the base coordinate of an arbitrary point at the longitudinal axis undergoing the deformation (u, w) is,

$$\begin{cases} x^0(x,t) = [x + u(x,t)]\cos\phi(t) - w(x,t)\sin\phi(t) \\ y^0(x,t) = [x + u(x,t)]\sin\phi(t) + w(x,t)\cos\phi(t) \end{cases} \tag{1}$$

consequently, the square of velocity of the point is ,

$$(\dot{x}^0)^2 + (\dot{y}^0)^2 = (\dot{u} - w\dot{\phi})^2 + [(u + x)\dot{\phi} + \dot{w}]^2 = \Delta_x^2 + \Delta_y^2$$

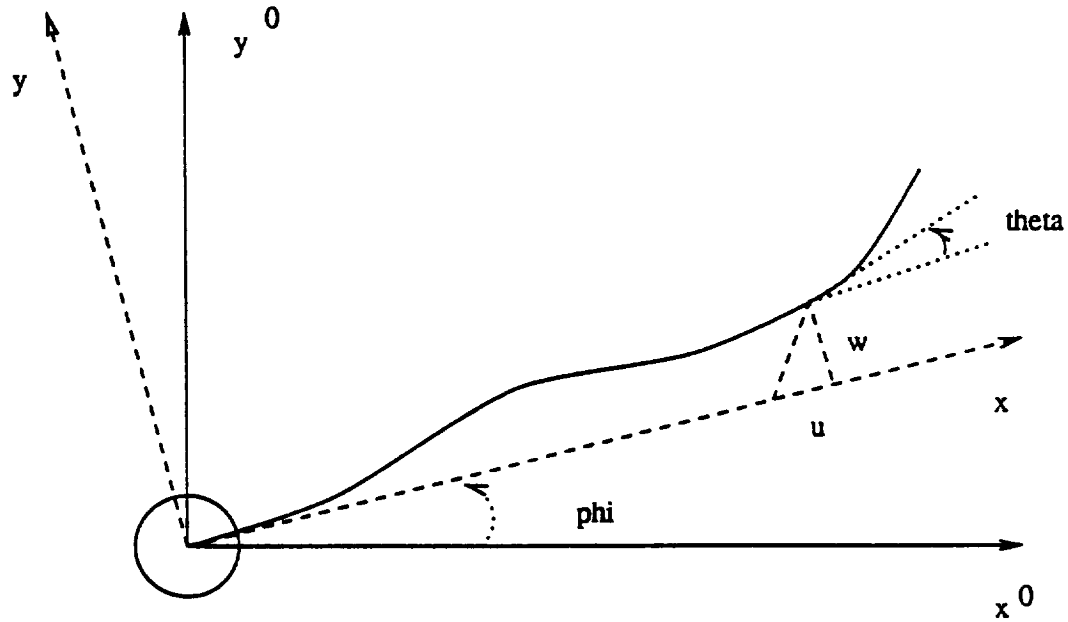

Figure 1: Cantilevered Beam attached to Rotating Hub

where

$$\boxed{\Delta_x = \dot{u} - w\dot{\phi}}, \qquad \boxed{\Delta_y = (u + x)\dot{\phi} + \dot{w}}.$$

As in the theory of bending of beams, we have used the normal plane assumption, that is, the entire transverse section of the beam remains planar and normal to the longitudinal axis of the beam after bending [11]. With this assumption, the following equation can be derived,

$$\frac{1}{\rho_d} = \frac{M}{EI} \tag{2}$$

where ρ_d denotes the radius of curvature of the deflected axis of the beam, M the corresponding external bending moment, and EI the flexural rigidity of the beam.

Let θ be the angle between the tangent of the deflected axis and x-axis. From geometry, it follows that

$$\frac{1}{\rho_d} = \frac{\partial \theta}{\partial x}. \tag{3}$$

Since the elongation of the longitudinal axis is assumed to be negligible, the following relation between θ and displacements u and w can be obtained,

$$\cos\theta = \frac{\partial u}{\partial x} + 1, \qquad \sin\theta = \frac{\partial w}{\partial x} \tag{4}$$

Since $u(0,t)$=0 and $w(0,t)$=0, u and w can be determined in terms of θ by

$$u(x,t) = \int_0^x \cos\theta(\xi,t)d\xi - x, \qquad w(x,t) = \int_0^x \sin\theta(\xi,t)d\xi. \tag{5}$$

A variational approach is used to derive the dynamical motion equations of flexible manipulators. To this end, we need the kinetic energy T and potential energy V of the system, and the nonconservative work W by the input torque τ applied on the hub:

$$
\begin{aligned}
T &= \frac{1}{2} I_H \dot{\phi}^2 + \frac{1}{2}\int_0^L \rho[(\dot{x}^0)^2 + (\dot{y}^0)^2]dx, \\
V &= \frac{1}{2}\int_0^L EI\left(\frac{\partial \theta}{\partial x}\right)^2 dx, \\
W &= \tau\phi.
\end{aligned}
$$

where ρ is the beam density (mass per unit length).

Hamilton's Principle [12] states that,

$$
\delta \int_a^b (T - V + W)dt = 0
$$

To carry out the variational calculation, we first note that

$$
\begin{aligned}
\frac{1}{2}\delta[(\dot{x}^0)^2 + (\dot{y}^0)^2] &= \Delta_x \delta\Delta_x + \Delta_y \delta \Delta_y \\
&= \Delta_x(\delta\dot{u} - w\delta\dot{\phi} - \dot{\phi}\delta w) + \Delta_y[\delta\dot{w} + (u+s)\delta\dot{\phi} + \dot{\phi}\delta u] \\
&= \dot{\overline{\Delta_x\delta u + [(u+s)\Delta_y - w\Delta_x]\delta\phi + \Delta_y\delta w}} + (\dot{\phi}\Delta_y - \dot{\Delta}_x)\delta u \\
&\quad + \dot{\overline{w\Delta_x - (u+s)\Delta_y}}\delta\phi - (\dot{\phi}\Delta_x + \dot{\Delta}_y)\delta w
\end{aligned}
$$

therefore,

$$
\delta\int_a^b T dt = -\int_a^b I_H\ddot{\phi}\delta\phi dt + \int_a^b\int_0^L [(\dot{\phi}\Delta_y - \dot{\Delta}_x)\delta u - \dot{\overline{(u+s)\Delta_y - w\Delta_x}}\delta\phi - (\dot{\phi}\Delta_x + \dot{\Delta}_y)\delta w]\rho dx dt.
$$

Note that all variations at a and b are zero. From Appendix I,

$$
\begin{aligned}
\int_0^L \Sigma(x)\delta u(x)dx &= -\int_0^L \sin\theta(\xi)\delta\theta(\xi)\int_\xi^L \Sigma(x)dx d\xi, \\
\int_0^L \Sigma(x)\delta w(x)dx &= \int_0^L \cos\theta(\xi)\delta\theta(\xi)\int_\xi^L \Sigma(x)dx d\xi.
\end{aligned}
$$

After substitution, we arrive at

$$
\begin{aligned}
\delta\int_a^b T dt &= -\int_a^b I_H\ddot{\phi}\delta\phi dt - \int_a^b\int_0^L \dot{\overline{(u+x)\Delta_y - w\Delta_x}}\rho dx \delta\phi dt \\
&\quad - \int_a^b\int_0^L\int_\xi^L [(\dot{\phi}\Delta_y - \dot{\Delta}_x)\sin\theta(\xi) + (\dot{\phi}\Delta_x + \dot{\Delta}_y)\cos\theta(\xi)]\delta\theta(\xi)\rho\, dx\, d\xi\, dt
\end{aligned}
$$

For the potential energy and external work, we have

$$
\delta\int_a^b V\,dt = -\int_a^b\int_0^L EI\frac{\partial^2\theta}{\partial x^2}\delta\theta\, dx\, dt \qquad \delta\int_a^b W\,dt = \int_a^b \tau\delta\phi\, dt.
$$

Substituting these variational expressions into Hamilton's Principle, we obtain the following governing integro-partial differential equations for dynamical motion of flexible manipulators,

$$EI\frac{\partial^2\theta}{\partial x^2} = \int_x^L \rho\left\{[\dot{\phi}\Delta_y(\xi) - \dot{\Delta}_x(\xi)]\sin\theta(x) + [\dot{\phi}\Delta_x(\xi) + \dot{\Delta}_y(\xi)]\cos\theta(x)\right\}d\xi \tag{6}$$

$$\tau = I_H\ddot{\phi} + \frac{d}{dt}\int_0^L \rho[(u+\xi)\Delta_y(\xi) - w\Delta_x(\xi)]d\xi \tag{7}$$

with the boundary conditions:

$$\theta(0,t) = 0; \qquad \theta'(L,t) = 0 \qquad x = L \tag{8}$$

The governing equations (6)–(7) and boundary conditions (8) constitute the nonlinear dynamical motion equations for one-link flexible manipulators.

3 Linearized Dynamical Motion Equations

Equations (6)–(8) are integro–differential equations which are difficult to work with directly. We adopt the approach of first linearizing the equation of motion and obtain the natural modes, and then use the natural modes as a basis of expansion for the general solution of the nonlinear equation. To this end, assume that the angle θ and *all* velocity quantities are small and all terms higher than second order are negligible.

$$\frac{1}{\rho_d} = \frac{\partial\theta}{\partial x} \approx \frac{\partial^2 w}{\partial x^2}.$$

Differentiating (6) with respect to x, and keeping only the linear terms, we obtain the Euler–Bernoulli model for the beam dynamics:

$$EI\frac{\partial^4 w}{\partial x^4} + \rho(x\ddot{\phi} + \ddot{w}) = 0, \tag{9}$$

with the boundary condition

$$w(0,t) = 0, \qquad w'(0,t) = 0, \qquad w''(L,t) = 0, \qquad w'''(L,t) = 0, \tag{10}$$

and the hub dynamics is given by

$$\tau - I_H\ddot{\phi} - \frac{d}{dt}\int_0^L x(x\dot{\phi} + \dot{w})\rho\, dx = 0. \tag{11}$$

Equations (9) and (11), and the corresponding initial and boundary conditions have been obtained by [3, 6]. The integral equation (11) can be reduced to an algebraic equation. This is accomplished by multiplying both sides of (9) by x and then integrating over $[0, L]$. The result is,

$$EI\frac{\partial^3 w}{\partial x^3}x\Big|_0^L - EI\frac{\partial^2 w}{\partial x^2}\Big|_0^L + \frac{d}{dt}\int_0^L x(x\dot{\phi} + \dot{w})\rho dx = 0.$$

After substitution, (11) can be written as,

$$\tau - I_H\ddot{\phi} + EIw''(0,t) = 0.$$

Let $v = w + x\phi$, equations (9) and (11) can be rewritten as,

$$EI\frac{\partial^4 v}{\partial x^4} + \rho\frac{\partial^2 v}{\partial t^2} = 0 \tag{12}$$

$$\tau - I_H\ddot{\phi} + EIv''(0,t) = 0. \tag{13}$$

The corresponding boundary conditions can be obtained from those for w and ϕ as

$$v(0,t) = 0, \quad v'(0,t) = \phi \qquad v''(L,t) = 0, \quad v'''(L,t) = 0. \tag{14}$$

Equations (12) and (13) are the most widely used governing equations for one-link flexible manipulators, [1, 5]. (However, the additional term $\rho x\ddot{\theta}$ appeared in [5] should be removed). It should be pointed out that in [1, 5], equation (13) had been actually considered as a boundary condition and the second equation (clamping condition) in boundary conditions (14) had not been included. The consequence of such practice is that w and ϕ can not be determined simultaneously since the system of equations becomes underconstrained. To solve w and ϕ simultaneously, we believe the clamping condition is required. This can also be justified by the following boundary variational condition resulted from Hamilton Principle:

$$EI\frac{\partial^2 w}{\partial x^2}\delta\frac{\partial w}{\partial x}\Big|_{x=0} = 0,$$

Therefore, either $\partial^2 w/\partial x^2 = 0$ or $\partial w/\partial x = 0$ has to be true at $x = 0$. Since it has been assumed that the flexible beam is fixed on the hub, the clamping condition holds. In [4], the correct boundary conditions (14) are included but the $v''(0,t)$ term in (13) was missing.

4 Discretization of Linearized Equation along Natural Modes

A common way to discretize the linearized partial differential equation (12)–(14) is to perform an eigenanalysis. To this end, consider a sinusoidal torque $\tau = \tau_o \sin(\omega t)$ applied at the hub. It is easy to show that the explicit solution of (12)–(14) can be obtained as

$$\phi(t) = \phi_o \sin(\omega t) \tag{15}$$

$$v(x,t) = v_o(x)\sin(\omega t) \quad . \tag{16}$$

Substituting the expression back into the partial differential equation (12) results in a fourth-order ordinary differential equation

$$EIv_o'''' - \rho\omega^2 v_o = 0 \quad . \tag{17}$$

The solution of this equation is given by (except for the rigid body, or the zero frequency solution)

$$v_o(x) = A\sin(kx) + B\sinh(kx) + C\cos(kx) + D\cosh(kx) \tag{18}$$

mode number	analytic (Hz)	experimental (Hz)
0	0	0
1	2.969	2.85
2	7.261	7.20
3	17.98	18.42
4	34.75	35.65
5	57.28	58.70
6	85.48	88.00
7	119.3	126.3
8	158.9	166.6
9	204.0	214.4

Table 1: Modal Frequency Comparison

where

$$EIk^4 = \rho\omega^2. \tag{19}$$

Substituting into Eq. (13) and (14) result in the following matrix equation for the unknown constants A, B, C, D, and ϕ_o:

$$\begin{bmatrix} 0 & 0 & 1 & 1 & 0 \\ k & k & 0 & 0 & -1 \\ -\mathrm{s} & \mathrm{sh} & -\mathrm{c} & \mathrm{ch} & 0 \\ -\mathrm{c} & \mathrm{ch} & \mathrm{s} & \mathrm{sh} & 0 \\ 0 & 0 & -k^2 & k^2 & \frac{I_H\omega^2}{EI} \end{bmatrix} \begin{bmatrix} A \\ B \\ C \\ D \\ \phi_o \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \frac{\tau_o}{EI} \end{bmatrix} \tag{20}$$

where we have used the short hand notation s for $\sin(kL)$, c for $\cos(kL)$, sh for $\sinh(kL)$, and ch for $\cosh(kL)$. Define the 5×5 matrix in the above equation as $\mathcal{A}$. Then the resonant frequencies are given by the solution of $\det \mathcal{A} = 0$ which can be further written as (as in [6])

$$\frac{I_H}{\rho}k^3(1 + \mathrm{c\ ch}) + (\mathrm{s\ ch} - \mathrm{c\ sh}) = 0. \tag{21}$$

Note the expression in the first set of parentheses equal to zero is the resonant frequency condition for a clamped–free beam and the expression in the second set of parentheses equal to zero is the resonant frequency condition for a pinned–free beam. The countably infinite number of solutions of (21) correspond to the resonant frequencies of the linearized rotating beam. In contrast to the assumed mode approach, there is *no* approximation in this expression. Therefore, fewer number of calculations need to be performed for a required number of modes, with higher accuracy.

When $\omega = 0$, which corresponds to the rigid body mode, the eigenfunction is $\psi_0(x) = cx$, where c is the normalization constant.

Table 1 lists the experimentally obtained modal frequencies for the first nine bending modes. They agree with the analytic prediction to within $\pm 5\%$.

Denote the nth solution, $n = 1, 2, \ldots$, by k_n, and the nth resonant frequency as ω_n. The vector $[A, B, C, D, \phi_o]^T$ that lies in the null space of $\mathcal{A}$ evaluated at k_n gives the eigenvector $\psi_n(x)$:

$$\psi_n(x) = A_n \sin(k_n x) + B_n \sinh(k_n x) + C_n \cos(k_n x) + D_n \cosh(k_n x) \tag{22}$$

where C_n is chosen as the arbitrary constant and

$$\begin{aligned} A_n &= \frac{\mathrm{c\ ch} + \mathrm{s\ sh} + 1}{-\mathrm{s\ ch} + \mathrm{c\ sh}} C_n \\ B_n &= \frac{2\rho}{I_H k_n^3} C_n - A_n \\ D_n &= -C_n \quad . \end{aligned} \tag{23}$$

From (13) and (14) with $\tau = 0$, ψ_n satisfies the following boundary conditions:

$$\begin{aligned} &\psi_n(0) = \psi_n'(0) = \psi_n''(L) = \psi_n'''(L) = 0 \\ &EI\psi_n''(0) = -I_H {\omega_n}^2 \psi_n'(0). \end{aligned} \tag{24}$$

The arbitrary constant can be chosen as a normalization constant. To derive the orthonormality condition, we substitute the eigenfunction $\psi_n(x)$ into (17) and take the $L_2[0, L]$ innerproduct with $\psi_m(x)$. After integration by parts, we obtain the orthonormality condition:

$$\langle \psi_n, \psi_m \rangle + \frac{I_H}{\rho} \psi_n'(0)\psi_m'(0) = a\delta_{nm} \tag{25}$$

where a is a constant that keeps the units consistent (therefore, a has the unit of distance cube), δ_{nm} is the Kronecker delta function. Note that a is usually set to be zero once a distance unit is chosen.

The normalized rigid body mode can be explicitly calculated

$$\psi_0(x) = \sqrt{a} \left(\frac{L^3}{3} + \frac{I_H}{\rho} \right)^{-\frac{1}{2}} x. \tag{26}$$

To discretize the linearized partial differential equation (12)–(14), the solution is expanded along the eigenfunctions:

$$v(x, t) = \sum_{n=0}^{\infty} q_n(t)\psi_n(x) \tag{27}$$

where $q_n(t)$ is called the modal amplitude function and $\psi_n(x)$ is called the mode shape. Substituting into the linearized PDE and apply integration by parts (see Appendix II), we have

$$\ddot{q}_n + {\omega_n}^2 q_n = \frac{\psi_n'(0)}{a\rho} \tau. \tag{28}$$

In the first order vectorial form, we have

$$\begin{bmatrix} \dot{q} \\ \ddot{q} \end{bmatrix} = \begin{bmatrix} 0 & I \\ -\Omega^2 & 0 \end{bmatrix} \begin{bmatrix} q \\ \dot{q} \end{bmatrix} + \begin{bmatrix} 0 \\ a^{-1}B \end{bmatrix} \tau \tag{29}$$

where q is an infinite dimensional column vector, 0 and I are infinite dimensional zero and identity matrices, respectively, Ω is a diagonal matrix with nth diagonal entry ω_n, and B is a column with nth element $\frac{\psi_n'(0)}{\rho}$.

Eq. (29) is of the form

$$A = \begin{bmatrix} 0 & I \\ -\Omega^2 & -D \end{bmatrix} \tag{30}$$

where D may be a visco–elastic damping operator (a 0 operator in the present case). The underlying space is $\mathbf{X} \triangleq \ell^2(0,\infty) \times \ell^2(0,\infty)$. The domain of the operator A is $\mathcal{D}(A) = \mathcal{D}(\Omega^2) \times \mathcal{D}(D)$ where

$$\mathcal{D}(\Omega^2) = \left\{ q \in \ell^2(0,\infty) : \sum_{n=0}^{\infty} {\omega_n}^2 q_n < \infty \right\}$$

and the domain of D is

$$\mathcal{D}(D) = \left\{ q \in \ell^2(0,\infty) : Dq \text{ is bounded.} \right\}.$$

5 Discretization of the Nonlinear Model

The eigenfunction expansion (27) can be used to discretize the nonlinear dynamical equation given in (6–8). To keep the equations tractable, we assume small bending, i.e., w' is small. Specifically, w and w' are assumed to be of the same order, say $\mathcal{O}(\epsilon)$, where ϵ is small. All velocities are assumed to be order 1. Terms with up to quadratic power of ϵ are kept in the expansion, so the equation of motion is valid up to the linear term in ϵ. This approach is the same as that in [9], except we do not assume small velocity. This results in nonzero centrifugal and Coriolis forces which were missing in the equation in [9]. The importance of these terms is in the preservation of the conservative property of the open loop system after the approximation. We shall see that this fact has an important consequence in the stability analysis. With the stated approximation, the kinetic energy becomes

$$T = \frac{1}{2}\rho \int_0^L \dot{v}^2\, dx + \frac{1}{2} I_H \dot{\phi}^2 + \frac{1}{2}\rho \int_0^L \left[\dot{u}^2 + w^2 \dot{\phi}^2 - 2w\dot{u}\dot{\phi} + 2u\dot{\phi}\dot{v} \right] \tag{31}$$

where u and $\dot{u}$ are approximated by

$$u(x,t) = -\frac{1}{2} \int_0^x w'(\xi,t)^2\, d\xi \tag{32}$$

$$\dot{u}(x,t) = -\int_0^x w'(\xi,t)\dot{w}'(\xi,t)\, d\xi. \tag{33}$$

The kinetic energy can be expanded along the natural modes of the linearized system. Then,

$$T = \frac{1}{2}\dot{q}^T M(q) q$$

where the mass matrix is

$$\begin{aligned} M(q) &= aI + \int_0^L A(x) q q^T A(x)\, dx + q^T A_1 q \Psi'(0)\Psi'^T(0) \\ &+ 2q^T \left[\int_0^L (\Psi(x) - x\Psi'(0)) A(x)\, dx \right] q \Psi'^T(0) - \Psi'(0) q^T \int_0^L A(x) q \Psi^T(x)\, dx \end{aligned} \tag{34}$$

q, $\Psi(x)$, $\Psi'(0)$ are infinite dimensional vectors for modal amplitudes, eigenfunctions evaluated at x, and the spatial derivative of eigenfunction evaluated at 0, respectively, and

$$\begin{aligned} A(x) &= \int_0^x (\Psi'(\xi) - \Psi'(0))(\Psi'(\xi) - \Psi'(0))^T \, d\xi \\ A_1 &= \int_0^L (\Psi(x) - x\Psi'(0))(\Psi(x) - x\Psi'(0))^T \, dx. \end{aligned}$$

The modal coordinate q is a generalized coordinate and the discretized kinetic energy can be used to find the Coriolis and centrifugal accelerations from

$$C(q,\dot{q})\dot{q} = \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}}\right) - \frac{\partial T}{\partial q}.$$

After some algebra, the following expression is obtained:

$$\begin{aligned} C(q,\dot{q})\dot{q} = \Big\{ \Big[& \int_0^L \left(\frac{L^2}{2} - \frac{x^2}{2}\right)(\Psi'(x) - \Psi'(0))(\Psi'(x) - \Psi'(0))^T \, dx \\ & - \int_0^L (\Psi(x) - x\Psi'(0))(\Psi(x) - x\Psi'(0))^T \, dx \Big] q\dot{q}^T \Psi'(0)\Psi'^T(0) \\ & + 2\Psi'(0)q^T \int_0^L (\Psi(x) - x\Psi'(0))(\Psi(x) - x\Psi'(0))^T \, dx \, \Psi'^T(0)\dot{q} \\ & - 2\dot{q}^T\Psi'(0) \int_0^L (\frac{L^2}{2} - \frac{x^2}{2})(\Psi'(x) - \Psi'(0))(\Psi'(x) - \Psi'(0))^T \, dx \, q\Psi'^T(0) \\ & + 2(\dot{q}^T \otimes I)\Big[\int_0^L \Big(\big(\int_x^L \Psi(\xi)\,d\xi\big) \otimes (\Psi'(x) - \Psi'(0))\Big)(\Psi'(x) - \Psi'(0))^T \, dx\Big] q\Psi'^T(0) \\ & - 2(\dot{q}^T \otimes I)\int_0^L \Big((\Psi'(x) - \Psi'(0)) \otimes \big(\int_x^L \Psi(\xi)\,d\xi\big)\Big)(\Psi'(x) - \Psi'(0))^T \, dx \, q\Psi'(0)^T \Big\}\dot{q} \end{aligned} \tag{35}$$

The discretized nonlinear dynamic equation accurate up to quadratic terms in w and w' can now be stated (generalizing (29)):

$$M(q)\ddot{q} + C(q,\dot{q})\dot{q} + a\Omega^2 q = \rho^{-1}\Psi'(0)\tau = B\tau. \tag{36}$$

This can be shown to correspond to the discretization of the following partial differential equation (generalizing (9)):

$$\begin{aligned} & \ddot{w} + x\ddot{\phi} + \frac{EI}{\rho}\frac{\partial^4 w}{\partial x^4} + w\dot{\phi}^2 + \dot{\phi}^2 \frac{d}{dx}\left[\left(\frac{L^2}{2} - \frac{x^2}{2}\right)w'\right] \\ & + 2\dot{\phi}\left\{\frac{d}{dx}[w'\int_x^L \dot{w}\,d\xi] - \int_0^x w'\dot{w}'\,d\xi\right\} + \left\{\frac{d}{dx}\left[w'\int_x^L\int_0^\xi \dot{w}'^2\,d\eta\,d\xi\right] - \int_0^x \dot{w}'^2\,d\xi\, w'\right\} \end{aligned} \tag{37}$$

The boundary conditions and the dynamic equation for ϕ are the same as before:

$$\tau - I_H\ddot{\phi} + EIw''(0,t) = 0 \tag{38}$$

$$w(0,t) = 0, \quad w'(0,t) = 0 \qquad w''(L,t) = 0, \quad w'''(L,t) = 0. \tag{39}$$

This model generalizes the one in [13] in which only the nonlinear term $\dot{\phi}^2\frac{d}{dx}\left[\left(\frac{L^2}{2} - \frac{x^2}{2}\right)w'\right]$ was included. This model also generalizes the one in [7], in which some but not all of the nonlinear terms are included, which implies the conservative nature of the open loop system is not preserved under the approximation.

6 Passivity, Control, and Stability Analysis

The centrifugal and Coriolis term $C(q,\dot{q})\dot{q}$ is related to the nonlinear mass matrix in an important way. Define M_D from the following relationship

$$\dot{M}(q,\dot{q})z = M_D(q,z)\dot{q}.$$

Then it is easy to show (same as in [14]) that one choice of C (it is not unique) is

$$C(q,\dot{q}) = M_D(q,\dot{q}) - \frac{1}{2}{M_D}^T(q,\dot{q}). \tag{40}$$

This relationship has been exploited extensively in the rigid robotics literature for stability analysis and control design, see for example [15, 14]. In fact, we can now show that joint angle proportional–derivative (PD) control is a stabilizing control law. Consider the following Lyapunov function candidate

$$V(q,\dot{q}) = \frac{1}{2}\dot{q}^T M(q)\dot{q} + a\frac{1}{2}q^T\Omega^2 q + \frac{1}{2}q^T BK_pB^T q \tag{41}$$

The derivative of V along the solution is

$$\begin{aligned} \dot{V}(q,\dot{q}) &= \dot{q}^T(-a\Omega^2 q + B\tau - C(q,\dot{q})\dot{q} + \frac{1}{2}\dot{M}(q,\dot{q}) + a\Omega^2 q) \\ &= \dot{q}^T B\tau. \end{aligned}$$

Note that the contribution due to $C(q,\dot{q})$ drops out due to the structure given in (40). The above energy Lyapunov analysis confirms the fact that the map from τ to $B^T\dot{q}$ (i.e., the joint angular velocity) is passive which is also the well–known sensor/actuator colocation condition. Note that if nonlinear terms are only retained in M but not in C as in [9] (i.e., setting it to zero, by assuming small $\dot{q}$), there would be an $\dot{M}$ term in $\dot{V}$, thus the passivity property would be destroyed. If τ is chosen as a simple joint angle PD control law

$$\tau = -K_pB^T(q - q_{des}) - K_vB^T\dot{q}, \tag{42}$$

where q_{des} is chosen to satisfy

$$\begin{aligned} B^T q_{des} &= \phi_{des} \\ \Omega q_{des} &= 0 \end{aligned}$$

simultaneously. It is possible to choose such q_{des} since $\begin{bmatrix} B^T \\ \Omega \end{bmatrix}$ is onto (note that the first component of B is $\rho^{-1}\left(\frac{L^3}{3} + \frac{I_H}{\rho}\right)$ which is non–zero).

With the joint angle PD control, the closed loop system is stable from the fact that $\dot{V} = -K_v(B^T\dot{q})^2 \leq 0$. Since $\dot{V} \leq 0$ implies all trajectories are uniformly bound in t, by [16], the joint angular velocity $B^T\dot{q}(t)$ tends to zero as $t \to \infty$. From the governing equation (36), all higher derivatives q are uniformly bounded. Using Proposition 1 in [17], it follows that $B^T\frac{d^k q(t)}{dt^k}$ converges to zero in norm for $k \geq 0$.

At this point, we revert to a local analysis for the linearized system, i.e., consider a neighborhood of the zero equilibrium where $\dot{q}$ is sufficiently small in which asymptotic convergence for equation (29) implies asymptotic convergence for the nonlinear system described (36). If only a finite number of modes is undamped, and the damping operator D is bounded relative to Ω^2, the closed loop infinitesimal generator (which is of the same form as A in (30) except D is replaced by $D + BK_vB^T$ and Ω^2 by $\Omega^2 + BK_pB^T$) has compact resolvent [18, Section 3.4] which implies that all bounded trajectories are precompact [19, Theorem 5.2]. Hence, the invariance principle can be applied to the closed loop infinite–dimensional system, i.e., all trajectories converge to the largest invariant set in

$$\mathcal{Q} \triangleq \left\{(q, \dot{q}) : B^T\dot{q} = 0\right\}. \tag{43}$$

If $B^T\dot{q}$ is detectable, then the largest invariant set in $\mathcal{Q}$ is just the origin and the zero equilibrium is asymptotically stable. When does the detectability condition hold? Without loss of generality, assume the first N modes are undamped. Therefore, to check detectability, we only need to check the observability of the first N modes. By forming the observability matrix, it follows that if the $N \times N$ matrix

$$\mathcal{O} \triangleq \begin{bmatrix} B^T \\ B^TK^2 \\ \vdots \\ B^TK^{2(N-1)} \end{bmatrix},$$

where $K \triangleq \Omega^2 + BK_pB^T$, is invertible, then the observability condition holds and the flexible beam, with N unstable modes, is stabilized with just joint angle PD feedback. The condition that $\mathcal{O}$ is invertible is very mild, in fact, it is exactly the same as the stabilizability condition from joint torque.

The above discussion can also be viewed from a general passivity perspective (this line of reasoning was originally proposed for flexible joint control [20]). This discussion is best understood through a number of steps:

1. First consider just the joint angular position feedback $\tau = \tau_1 - K_pB^T\Delta q$. Then the map from τ_1 to $B^T\dot{q}$ is passive by following the above Lyapunov argument.

2. Let $\mathcal{C}$ be *any* strictly passive map that takes $B^T\dot{q}$ to τ_1:

   $$\tau_1 = \tau_2 - \mathcal{C}(B^T\dot{q}).$$

   The constant gain K_v feedback is a special case.

3. By the Passivity Theorem [21], the map that takes τ_2 to $B^T\dot{q}$ is L_2–input/output stable.

4. If the closed loop system is detectable with respect to $B^T\dot{q}$ and stabilizable with respect to τ_2, then the system is internally asymptotically stable.

If the feedback $\mathcal{C}$ is restricted to be linear, then it must be strictly positive real [22, 23]. An open and very interesting problem is on choosing a strictly positive real $\mathcal{C}$ so that some performance measure (e.g., H_∞ norm of some input/output pair for the linearized system) is optimized. Another implication of the above discussion is that any feedback controller from $B^T\dot{q}$ to τ_1 which is itself stable can be "robustified" by adding in a suitable amount of constant gain feedback. Sample data system can be included in this discussion by including the sampler and zero–order–hold in the consideration of passivity for the feedback system.

7 Concluding Remarks

A nonlinear dynamical model for one-link flexible manipulators undergoing large deformation has been developed by using Hamilton's Principle. The governing motion equations in this model are two highly coupled integro-partial differential equations. The paper has demonstrated that various types of simplified motion equations can be derived systematically from nonlinear equations by assuming small bending and then small velocity. It has also been argued that the clamping boundary condition at the hub end should be specified explicitly in order to determine the beam deformation and hub rotation simultaneously. The linearized equation is used to derive the modal frequencies and mode shapes which are in turn used to discretize the small bending nonlinear equation. A stability analysis using the passivity property of this equation is performed to show that joint PD control is globally stabilizing.

Acknowledgment

The authors would like to thank Mr. L. Lanari for helpful and stimulating discussions and the careful reading of the manuscript. This work was supported in part by NASA Center for Intelligent Robotic Systems for Space Exploration under Grant No.NAGW–1333, New York State Center for Advanced Technology in Robotics and Automation, and National Science Foundation under Grant No.MSS–8910437.

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Appendix I

By Dirichlet formula:

$$\int_0^L \int_0^x f(x,\xi)d\xi dx = \int_0^L \int_\xi^L f(x,\xi)dxd\xi.$$

Since,

$$\delta u(x,t) = -\int_0^x \sin\theta(\xi,t)\delta\theta(\xi)d\xi, \qquad \delta w(x,t) = \int_0^x \cos\theta(\xi,t)\delta\theta(\xi)d\xi$$

therefore,

$$\begin{aligned}
\int_0^L \Sigma(x)\delta u(x)dx &= -\int_0^L \Sigma(x)\int_0^x \sin\theta(\xi)\delta\theta(\xi)d\xi dx = -\int_0^L \sin\theta(\xi)\delta\theta(\xi)\int_\xi^L \Sigma(x)dxd\xi, \\
\int_0^L \Sigma(x)\delta w(x)dx &= \int_0^L \Sigma(x)\int_0^x \cos\theta(\xi)\delta\theta(\xi)d\xi dx = \int_0^L \cos\theta(\xi)\delta\theta(\xi)\int_\xi^L \Sigma(x)dxd\xi.
\end{aligned}$$

Appendix II

Innerproduct both sides of (12) with ψ_m and substitute in the expansion (27) and boundary condition (24) , we have

$$\begin{aligned}
& \sum_{n=0}^\infty \ddot{q}_n \langle\psi_n,\psi_m\rangle + \frac{EI}{\rho}\Big(v'''\psi_m\Big|_0^L - v''\psi_m'\Big|_0^L + v'\psi_m''\Big|_0^L - v\psi_m'''\Big|_0^L + \langle v,\psi_m''''\rangle\Big) \\
= & \sum_{n=0}^\infty \ddot{q}_n \langle\psi_n,\psi_m\rangle + \frac{EI}{\rho}\Big(v''(0,t)\psi_m'(0) - v'(0,t)\psi_m''(0)\Big) + {\omega_m}^2\langle v,\psi_m\rangle \\
= & \sum_{n=0}^\infty (\ddot{q}_n + \omega_m q_n)\langle\psi_n,\psi_m\rangle + \frac{I_H}{\rho}\psi_m'(0)\sum_{n=0}^\infty(\ddot{q}_n + \omega_m q_n)\psi_n'(0) - \frac{\tau}{\rho} \\
= & a(\ddot{q}_m + {\omega_m}^2 q_m) - \frac{\tau}{\rho}
\end{aligned}$$

where the last expression is obtained by using the orthonormality condition (25).